New Developments in Belief Revision

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Chapter 6
Non-monotonic Reasoning - Conditionals
BR and Non-monotonicity

Today’s topic

In the AGM approach we assume that the underlying consequence operator satisfies some properties. In particular, it is assumed that it is Tarskian and Compact.

• Are these constraints essential for developing and AGM-style analysis?

• Today we consider dropping one of the property that a consequence operator needs to satisfy to be Tarskian: Monotonicity.
AGM Assumptions

AGM made some assumptions about the underlying logic $\langle L, Cn \rangle$:

- **Language**: closed under propositional operators.

- **Consequence operator**:

  1. **Tarskian**

     - **Monotonicity**: if $A \subseteq B$ then $Cn(A) \subseteq Cn(B)$

     - **Idempotence**: $Cn(A) = Cn(Cn(A))$

     - **Inclusion**: $A \subseteq Cn(A)$
AGM Assumptions

2. AGM Assumptions:

• **Deduction:** \( \beta \in Cn(A \cup \{\alpha\}) \) iff \( (\alpha \rightarrow \beta) \in Cn(A) \)

• **Supraclassicality:** if \( \alpha \in Cl(A) \) then \( \alpha \in Cn(A) \)

• **Compactness:** if \( \alpha \in Cn(A) \), then \( \alpha \in Cn(A') \) for some finite \( A' \subseteq A \)

• **Disjunction in the premises:** \[ \gamma \in Cn(A \cup \{\alpha\}) \quad \gamma \in Cn(A \cup \{\beta\}) \]
  \[ \gamma \in Cn(A \cup \{\alpha \lor \beta\}) \]
AGM Assumptions

We may need an analysis of belief change for logics that do not satisfy the above prerequisites.

➡ What happens if we drop some of them?
Can we still develop an AGM-style analysis of belief change?

Today we consider dropping one of the Tarskian properties:

Monotonicity
Non-monotonicity

**Monotonic reasoning**

\[ A \subseteq B \implies Cn(A) \subseteq Cn(B) \]

Certainty of the conclusions.

Whatever new information we add to our knowledge base, our previous conclusions remain valid.

**Non-monotonic reasoning**

\[ A \subseteq B \nRightarrow Cn(A) \nsubseteq Cn(B) \]

We draw also tentative, defeasible conclusions.

New information could put in jeopardy our previous conclusions.
Non-monotonicity

Recent work in revision of non-monotonic theories:

➡ Answer Set Programming

➡ Conditional Reasoning

Today we will focus on conditional reasoning:

➡ Its relation with belief revision

➡ Known issues

➡ Recent characterisations of BR for conditional reasoning

And at the end also have a look at ASP.
Non-monotonicity

Example

Remember the KB we saw in the first lecture?

- Sweden is a part of Europe
- All European swans are white
- The bird caught in the trap is a swan
- The bird caught in the trap is from Sweden

From this we can derive
- The bird caught in the trap is white

If we are informed that the caught bird is a black swan, we need to revise our KB in order to preserve consistency.
Non-monotonicity

Example

Consider the following slightly modified KB:

- *Sweden is a part of Europe*
- *Typically, European swans are white*
- *The bird caught in the trap is a swan*
- *The bird caught in the trap is from Sweden*

From such a KB we can conclude that
- *Presumably, the bird caught in the trap is white*

Such a conclusion is just tentative.

We are informed that the swan is black

We drop the presumptive conclusion, we do not need to make changes to the KB, since it admits exceptions to the second statement.
Non-monotonicity

There is a connection between Belief Revision and Non-monotonic Reasoning

• Both are aimed at managing potential conflicts among pieces of information
  • Non-monotonic reasoning can manage conflicting information
  • Still, it is possible to have inconsistencies also in non-monotonic KBs

Example

Consider again the KB:

• *Sweden is a part of Europe*
• *Typically, European swans are white*
• *The bird caught in the trap is a swan*
• *The bird caught in the trap is from Sweden*

We are informed that
• *Typically, European swans are blue*

This is a conflict that is problematic also for non-monotonic systems.
Non-monotonicity

Assume we are facing conflicting pieces of information: should such a conflict be managed by the non-monotonic machinery or by some belief change operator?

How to characterise such belief change operators for non-monotonic reasoning?
Conditional Reasoning

We will consider conditional reasoning, where conditionals

\[ \alpha \Rightarrow \beta \]

are open to multiple interpretations, generally consistent with the existence of exceptions:

It is possible for the semantics to satisfy

\[ \alpha \Rightarrow \beta \ (\text{e.g.: typically, swans are white}) \]

while still allowing instantiations of \( \alpha \land \neg \beta \) (e.g.: there are black swans)
Conditional Reasoning

In Conditional Reasoning our KB is composed of a set of conditionals, and our reasoning procedures allow us to derive new conditionals.

Entailment operations \( Cn \) on the conditionals can be defined, satisfying specific structural properties, appropriate for the interpretation we give to the conditionals.

Example
Some examples of properties we could impose on the entailment operation:

**Conjunction on the right (AND)**

\[
\alpha \Rightarrow \beta \quad \alpha \Rightarrow \gamma \\
\frac{\alpha \Rightarrow \beta \land \gamma}{\alpha \Rightarrow \beta \land \gamma}
\]

**Disjunction on the left (OR)**

\[
\frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \lor \beta \Rightarrow \gamma}
\]
Conditional Reasoning

Example

These properties can be more or less desirable w.r.t what notion the conditionals model.

For example, (AND):

**Conditional obligations**

\[
\begin{align*}
\text{road\_accident} &\Rightarrow \neg \text{leave} & \text{road\_accident} &\Rightarrow \text{call\_police} \\
\phantom{\text{road\_accident}} &\Rightarrow \neg \text{leave} \land \text{call\_police}
\end{align*}
\]

**Conditional desires**

\[
\begin{align*}
\text{evening} &\Rightarrow \text{netflix\_at\_home} & \text{evening} &\Rightarrow \text{go\_to\_party} \\
\phantom{\text{evening}} &\Rightarrow \text{netflix\_at\_home} \land \text{go\_to\_party}
\end{align*}
\]
Conditional Reasoning

Example

These properties can be more or less desirable w.r.t what notion the conditionals model.

For example, (AND):

Conditional obligations

\[
\text{road.accident} \Rightarrow \neg \text{leave} \quad \text{road.accident} \Rightarrow \text{call.police}
\]

\[
\text{road.accident} \Rightarrow \neg \text{leave} \land \text{call.police} \quad \text{ok!}
\]

Conditional desires

\[
\text{evening} \Rightarrow \text{netflix.at.home} \quad \text{evening} \Rightarrow \text{go.to.party}
\]

\[
\text{evening} \Rightarrow \text{netflix.at.home} \land \text{go.to.party}
\]
Conditional Reasoning

Example

These properties can be more or less desirable w.r.t what notion the conditionals model.

For example, (AND):

Conditional obligations

\[
\text{road.accident} \Rightarrow \neg \text{leave} \quad \text{road.accident} \Rightarrow \text{call.police} \quad \text{ok!}
\]

\[
\text{road.accident} \Rightarrow \neg \text{leave} \land \text{call.police}
\]

Conditional desires

\[
\text{evening} \Rightarrow \text{netflix.at.home} \quad \text{evening} \Rightarrow \text{go.to.party} \quad \text{not ok!}
\]

\[
\text{evening} \Rightarrow \text{netflix.at.home} \land \text{go.to.party}
\]
Conditional Reasoning

For non-monotonic conditionals, the following does not hold:

\[ \frac{\alpha \Rightarrow \beta}{\alpha \land \gamma \Rightarrow \beta} \]

Conditionals like \(\text{bird} \Rightarrow \text{fly}\) and \(\text{penguin} \land \text{bird} \Rightarrow \neg \text{fly}\) can coexist consistently.

Note:
non-monotonicity conditional \(\neq\) non-monotonicity entailment operator

Monotonic entailment operator \(Cn\):

\[ \text{If } \alpha \Rightarrow \beta \in Cn(K) \text{ then } \alpha \Rightarrow \beta \in Cn(K \cup \{\gamma \Rightarrow \delta\}) \]

It is compatible with non-monotonic conditionals.
Conditional Reasoning

A popular semantics for non-monotonic conditionals $\alpha \Rightarrow \beta$ : preferential semantics.

Interpretations $M = (W, \prec)$:

- $W$ is a (multi)set of possible worlds (propositional valuations)
- $\prec$ is a preference relation defined over $W$:
  - transitive, asymmetric, and smooth

Smoothness: for every $\alpha$, if $\text{Mod}(\alpha) \neq \emptyset$ then $\min(\text{Mod}(\alpha)) \neq \emptyset$, where

$$\min(\text{Mod}(\alpha)) = \{ w \in \text{Mod}(\alpha) \mid \exists v \in w \text{ s.t. } v \in \text{Mod}(\alpha) \text{ and } v \prec w \}$$

$w \prec v$ is read as ‘the situation described by $w$ is preferred to the situation described by $v$’
Conditional Reasoning

A conditional $\alpha \Rightarrow \beta$ is satisfied by an interpretation $M = (W, <)$ ($M \models \alpha \Rightarrow \beta$) if the preferred worlds satisfying $\alpha$ satisfy also $\beta$. That is,

$$\min(\text{Mod}(\alpha)) \subseteq \text{Mod}(\beta)$$

Depending on the interpretation we give to the relation $<$, the conditionals $\alpha \Rightarrow \beta$ have been interpreted in various ways. For example:

- **Expectations**: “Typically, if $\alpha$ then $\beta$“.

- **Obligations**: “If $\alpha$, then it ought to be $\beta$”.

- **Counterfactual/subjunctive conditionals**: “If $\alpha$ were the case, then $\beta$ would have been the case too”.

Conditional Reasoning

An entailment operator modelled using preferential semantics is characterised by the satisfaction of the preferential properties:

- **Reflexivity**
  \[
  (\text{REF}) \quad \alpha \Rightarrow \alpha
  \]

- **Cut (Cumulative Trans.)**
  \[
  (\text{CT}) \quad \frac{\alpha \Rightarrow \beta \quad \alpha \land \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}
  \]

- **Cautious Monotony**
  \[
  (\text{CM}) \quad \frac{\alpha \Rightarrow \beta \quad \alpha \Rightarrow \gamma}{\alpha \land \beta \Rightarrow \gamma}
  \]

- **Left Logical Equival.**
  \[
  (\text{LLE}) \quad \frac{\alpha \Rightarrow \gamma \quad \models \alpha \equiv \beta}{\beta \Rightarrow \gamma}
  \]

- **Right Weakening**
  \[
  (\text{RW}) \quad \frac{\alpha \Rightarrow \beta \quad \beta \models \gamma}{\alpha \Rightarrow \gamma}
  \]

- **Left Disjunction**
  \[
  (\text{OR}) \quad \frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \lor \beta \Rightarrow \gamma}
  \]
Conditional Reasoning

Theorem [Kraus et Al. (1990)]
A conditional entailment operator $Cn(.)$ is closed under the preferential properties iff, for every conditional KB $K$, $Cn(K)$ can be defined using a preferential model. That is,

$$Cn(K) = \{ \alpha \Rightarrow \beta \mid M \models \alpha \Rightarrow \beta \}$$

for some preferential model $M$.

Special case - Preferential Closure $Pr(.)$:

$$Pr(K) = \{ \alpha \Rightarrow \beta \mid M \models \alpha \Rightarrow \beta \text{ for all } M \text{ s.t. } M \models K \}$$

$Pr(K)$ is the smallest preferential closure containing $K$.

$\alpha \Rightarrow \beta \in Pr(K)$ iff $\alpha \Rightarrow \beta$ is derivable from $K$ using the preferential properties

$Pr(.)$ is Tarskian!
(and hence monotonic)
Conditional Reasoning

Let’s consider another property:

\[
\begin{align*}
\alpha \Rightarrow \beta & \quad \alpha \not\Rightarrow \neg \gamma \\
\alpha \land \gamma & \Rightarrow \beta
\end{align*}
\]

Rational Monotony

necessary for the satisfaction of important reasoning patterns, as

**Presumption of typicality:**
Given the information at our disposal, we assume we are in the most expected situation.

\[
\text{bird} \Rightarrow \text{fly} \quad \text{bird} \not\Rightarrow \neg \text{sparrow}
\]

\[
\text{bird} \land \text{sparrow} \Rightarrow \text{fly}
\]
Conditional Reasoning

The entailment operators aimed at modelling some kind of presumptive reasoning are usually non-monotonic (and satisfy (RM)).

• I know that typically birds fly \((\text{bird} \Rightarrow \text{fly})\)

• I hear about some ‘Dodo’ bird, but I know nothing about it. So, I am not aware whether it is an atypical bird \((\text{bird} \not\Rightarrow \neg\text{dodo})\)

• With this information, I presume that dodos behave like normal birds \((\text{bird} \land \text{dodo} \Rightarrow \text{fly})\)

• Later I am informed that dodos are extinct, and that actually they were very strange birds. Not really a typical bird \((\text{bird} \Rightarrow \neg\text{dodo})\)

• With this new piece of information, I can to drop the previous conclusion, still satisfying (RM) \((\text{bird} \land \text{dodo} \not\Rightarrow \text{fly})\)

\[
dodo \land \text{bird} \Rightarrow \text{fly} \in Cn(\{\text{bird} \Rightarrow \text{fly}\})
\]

\[
dodo \land \text{bird} \Rightarrow \text{fly} \notin Cn(\{\text{bird} \Rightarrow \text{fly}, \text{bird} \Rightarrow \neg\text{dodo}\})
\]
Conditional Reasoning

Ranked interpretation $R = (W, <)$:

$R$ is a preferential interpretation and $<$ satisfies modularity:

If $x < y$ then either $z < y$ or $x < z$

Theorem [Lehmann and Magidor (1992)]

A conditional entailment operator $\text{Cn}(.)$ is closed under the preferential properties + (RM) iff, for every conditional KB $K$, $\text{Cn}(K)$ can be defined using a ranked model. That is,

$$\text{Cn}(K) = \{ \alpha \Rightarrow \beta \mid M \models \alpha \Rightarrow \beta \}$$

for some ranked model $R$. 
Two sides of the same coin

Remember the semantic characterisation of AGM revision?

It was built using a specific class of preferential models, the ones in which $\prec$ is modular.

The worlds in the yellow part define $K \star \alpha$
The yellow part corresponds to $\min(\text{Mod}(\alpha)) <$. So, there is a strong correspondence: in the same ranked model, $\alpha \Rightarrow \beta$ holds iff $\beta \in K \star \alpha$. 

**Two sides of the same coin**
Two sides of the same coin

There is the possibility of representing revision policies via subjunctive conditionals:

\[ \beta \in K \star \alpha \iff \alpha \Rightarrow \beta \]

That is, the revision policies can be represented via conditionals interpreted as “if \( \alpha \) were the case, then \( \beta \) would hold”.

**Question:** Can we extend the revision operators to a language containing the correspondent conditionals?

Implementing such a step would allow also to revise the revision policies.
Two sides of the same coin

Answer: No!
Or at least not easily.

Gärdenfors’ Impossibility result [Gärdenfors (1988)],
here in Rott’s version [Rott (1989)]:

Assume a logic \( \langle L, Cn \rangle \) where:

- **Language** \( L \): Propositional language + conditionals \( \alpha \Rightarrow \beta \)
- **Operator** \( Cn(.) \): operator for \( L \) s.t. it corresponds to propositional logic for the propositional fragment (and it is monotonic over the entire \( L \))

**Belief Revision Model** \( \langle K, \ast \rangle \): \( K \) is a set of \( Cn \)-theories (closed under propositional expansion) and \( \ast \) is a revision operation on the theories \( K \) in \( K \) satisfying:

\[
\begin{align*}
(\ast & \ 2) \text{ If } \alpha \notin Cn(\emptyset), \text{ then } \alpha \in K \ast \alpha (\text{Success}) \\
(\ast & \ 4) \text{ If } \neg \alpha \notin K, \text{ then } Cn(K \cup \{\alpha\}) \subseteq K \ast \alpha (\text{Vacuity}) \\
(\ast & \ 6) \text{ If } \neg \alpha \notin Cn(\emptyset), \text{ then } \bot \notin K \ast \alpha (\text{Consistency}) \\
\text{(GRT) } \beta \in K \ast \alpha \text{ iff } \alpha \Rightarrow \beta \in K (\text{Ramsey Test})
\end{align*}
\]
Two sides of the same coin

Let $B$ be a theory in $K$, and let $\alpha, \beta$ be two contingent propositions, that is, s.t.

$$\alpha \notin Cn(\emptyset); \neg \alpha \notin Cn(\emptyset); \beta \notin Cn(\emptyset); \neg \beta \notin Cn(\emptyset)$$

and such that $\{\alpha \lor \beta, \neg \alpha \lor \beta, \alpha \lor \neg \beta, \neg \alpha \lor \neg \beta\} \cap B = \emptyset$

It turns out that $\bot \in Cn(B \cup \{\alpha\}) \star \neg \alpha$

In contradiction with

$(\star 6)$ If $\neg \alpha \notin Cn(\emptyset)$, then $\bot \notin K \star \alpha$ (Consistency)

A lot has been discussed about the implications of Gardenfors’ result, and its hidden assumptions.

There have been some interesting proposals about the revision of conditionals respecting the Ramsey test, but avoiding the impossibility result.

All the discussion was focused on the Ramsey test and the subjunctive interpretation of conditionals
BR of non-monotonic KBs

Does Gardenfors’ result prevent the application of an AGM approach to non-monotonic preferential reasoning?

Preferential conditionals have also other interpretations beyond the subjunctive one. Belief change is interesting also in a non-monotonic conditional framework.

Example

We have a knowledge base $B$ containing the following information:

- vertebrate red blood cells have a nucleus ($v \rightarrow n$);
- avian red blood cells are vertebrate red blood cells ($a \rightarrow v$);
- mammalian red blood cells are vertebrate red blood cells ($m \rightarrow v$);
- mammalian red blood cells don't have a nucleus ($m \rightarrow \neg n$).

We must conclude that mammalian red blood cells do no exist ($m \rightarrow \bot$).
BR of non-monotonic KBs

Case 1.

We know that mammalian red blood cells exist, and we want to enforce such information ($m \rightarrow \bot$ should be contracted).

- In classical monotonic belief change: the contraction of $m \rightarrow \bot$ results into the elimination of some piece of information, for example $v \rightarrow n$;

- It could be preferable to weaken $v \rightarrow n$ into its defeasible version $v \Rightarrow n$ (vertebrate red blood cells usually have nucleus).

- The non-monotonic inference machinery will take care of treating $m$ as an exceptional subclass of $v$.

$$B' = \{ v \Rightarrow n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \}$$
Case 2.

Assuming our non-monotonic machinery is well-behaved,

- from $B'$ we can conclude that avian red blood cells presumably have a nucleus ($a \Rightarrow n$)
  \[ B' = \{ v \Rightarrow n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \} \]

- But we are informed that avian red blood cells usually do not have a nucleus ($a \Rightarrow \neg n$). Since $a \Rightarrow n$ was a presumptive conclusion, the non-monotonic entailment relation should take care of eliminating such a conclusion once faced with conflicting evidence ($a \Rightarrow \neg n$).

- The introduction of $a \Rightarrow \neg n$ should correspond to a simple addition:
  \[ B'' = \{ v \Rightarrow n, a \Rightarrow \neg n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \} \]
Case 3.

\[ B'' = \{ v \Rightarrow n, a \Rightarrow \neg n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \} \]

We are then informed that \( a \Rightarrow n \).

But, even in the most trivial non-monotonic reasoning, \( B'' \models a \Rightarrow \neg n \)

We have now a choice:

- If we are interested only in preserving logical consistency (avoid \( \top \Rightarrow \bot \)), then we can simply add \( a \Rightarrow n \) to \( B'' \), and conclude \( a \Rightarrow \bot \).
- If we want to preserve coherence, then we have to "re-adjust" the KB to avoid \( a \Rightarrow \bot \).

As it is intended in the field of logic-based ontologies, a KB is coherent if every class (i.e., atomic proposition) \( a \) that has been introduced in the language can in principle be populated (we cannot conclude \( a \Rightarrow \bot \)).
BR of non-monotonic KBs

There are two critical points in modelling belief change for conditional non-monotonic reasoning:

1. Revising, are we interested in preserving consistency or coherence?

   • We want to add to our KB a conditional \( \alpha \Rightarrow \beta \). Do we consider a potential conflict if the addition of \( \alpha \Rightarrow \beta \) enforces the derivation of \( \top \Rightarrow \bot \) (logical inconsistency) or it is sufficient the derivation of \( \alpha \Rightarrow \bot \) (incoherence)?

This is a contextual issue, associated to the domain we are modelling.
BR of non-monotonic KBs

2. Management of the potential conflicts.

- We have a non-monotonic consequence operator $Cn$ and a conditional base $K$. Let $\alpha \implies \neg \beta \in Cn(K)$. We receive the information $\alpha \implies \beta$, that is in conflict with $Cn(K)$. We need to know whether $\alpha \implies \neg \beta$ is a necessary or a defeasible consequence of $K$. In the former case, we have a conflict and we need to revise the base (Case 3 of the example), in the latter there is no need of actual revision, since the non-monotonic machinery will eliminate the conflict (Case 2 of the example).
BR for preferential conditionals

This second point is a formal question: given a non-monotonic closure $C_n$ and a conditional base $K$, which conditionals in $C_n(K)$ are a necessary consequence of $C_n$. Monotonicity gives us the answer.

A closure operator $Cl$ is called the monotonic core of a non-monotonic closure $C_n$ if, for every conditional base $B,B'$,

(i) $B \subseteq B'$ implies $Cl(B) \subseteq Cl(B')$;

(ii) $Cl(B) \subseteq C_n(B)$;

(iii) for every closure operator $Cl'$ satisfying (i) and (ii), $Cl'(B) \subseteq Cl(B)$.

Given a non-monotonic entailment relation, the existence of a monotonic core needs to be proved.
BR for preferential conditionals

We consider the class of **supra-preferential cumulative operators** $Cn$:

- **Supra-preferential**:
  - $Cn$ is closed under the preferential properties
  - If a set of conditionals $K$ has a model ($T \Rightarrow \bot \notin Pr(K)$), then $T \Rightarrow \bot \notin Cn(K)$ (consistency preservation)

- **Cumulative**:
  - if $B \subseteq B' \subseteq Cn(B)$, then $Cn(B') = Cn(B)$

This covers an ample class of non-monotonic operator $Cn$ that are definable using preferential semantics.
BR for preferential conditionals

Proposition [Casini & Meyer (2017)]

Given a supra-preferential closure operator $Cn$, its monotonic core is the preferential closure $Pr$.

Characterising belief revision for supra-preferential operators:

1. Model it for the monotonic core (preferential closure $Pr$).

2. Then model it for the non-monotonic operator $Cn$. 
Contraction and preferential closure

Translation of the basic AGM contraction postulates in the conditional framework:

\[
\begin{align*}
(\div 1) \quad & K \div \alpha = Cn(K \div \alpha) \\
(\div 2) \quad & K \div \alpha \subseteq K \\
(\div 3) \quad & \text{If } \alpha \not\in Cn(K), \text{ then } K \div \alpha = K \\
(\div 4) \quad & \text{If } \alpha \not\in Cn(\emptyset), \text{ then } \alpha \not\in K \div \alpha \\
(\div 5) \quad & \text{If } \alpha \leftrightarrow \beta \in Cn(\emptyset), \text{ then } K \div \alpha = K \div \beta \\
(\div 6) \quad & K \subseteq Cn((K \div \alpha) \cup \{\alpha\})
\end{align*}
\]

\[
\begin{align*}
(-1) \quad & K_{\alpha \Rightarrow \beta} = Pr(K_{\alpha \Rightarrow \beta}) \\
(-2) \quad & K_{\alpha \Rightarrow \beta} \subseteq K \\
(-3) \quad & \text{If } \alpha \Rightarrow \beta \not\in Pr(K), \text{ then } K_{\alpha \Rightarrow \beta} = K \\
(-4) \quad & \text{If } \alpha \Rightarrow \beta \not\in Pr(\emptyset), \text{ then } \alpha \Rightarrow \beta \not\in K_{\alpha \Rightarrow \beta} \\
(-5) \quad & \text{If } \alpha \Rightarrow \beta \equiv_{Pr} \alpha' \Rightarrow \beta', \text{ then } K_{\alpha \Rightarrow \beta} = K_{\alpha' \Rightarrow \beta'} \\
(-6) \quad & K \subseteq Pr(K_{\alpha \Rightarrow \beta} \cup \{\alpha \Rightarrow \beta\})
\end{align*}
\]
Contraction and preferential closure

Remember the partial meet contraction in AGM belief revision?

The remainder set $K \perp \alpha$ contains all the maximal subtheories of $K$ that do not contain $\alpha$

$$K \div \alpha = \bigcap \gamma(K \perp \alpha)$$

Working with preferential theories $K$, we can define the equivalent notion in the conditional framework:

- $K \perp (\alpha \Rightarrow \beta)$ is the set of the maximal (preferential) subtheories of $K$ that do not contain $\alpha \Rightarrow \beta$

- $\div$ is a partial meet contraction operator if it can be defined as $K^-_{\alpha \Rightarrow \beta} = \bigcap \gamma(K \perp (\alpha \Rightarrow \beta))$

where $\gamma$ behaves as in the propositional case:

- If $K \perp (\alpha \Rightarrow \beta) \neq \emptyset$, then $\emptyset \neq \gamma(K \perp (\alpha \Rightarrow \beta)) \subseteq K \perp (\alpha \Rightarrow \beta)$

- If $K \perp (\alpha \Rightarrow \beta) = \emptyset$, then $\gamma(K \perp (\alpha \Rightarrow \beta)) = \{K\}$
Contraction and preferential closure

Monotonic Core $Pr$.

Postulates for Contraction –

The postulates for contraction are as follows (where $\equiv_{Pr}$ refers to preferential equivalence):

- $(-1)$\hspace{1cm} $K_{\alpha \rightarrow \beta}^- = Pr(K_{\alpha \rightarrow \beta}^-)$  \hspace{1cm} (− closure)
- $(-2)$\hspace{1cm} $K_{\alpha \rightarrow \beta}^- \subseteq K$ \hspace{1cm} (− inclusion)
- $(-3)$\hspace{1cm} If $\alpha \Rightarrow \beta \notin Pr(K)$, then $K_{\alpha \rightarrow \beta}^- = K$ \hspace{1cm} (− vacuity)
- $(-4)$\hspace{1cm} If $\alpha \Rightarrow \beta \notin Pr(\emptyset)$, then $\alpha \Rightarrow \beta \notin K_{\alpha \rightarrow \beta}^-$ \hspace{1cm} (− success)
- $(-5)$\hspace{1cm} If $\alpha \Rightarrow \beta \equiv_{Pr} \alpha' \Rightarrow \beta'$, then $K_{\alpha \rightarrow \beta}^- = K_{\alpha' \rightarrow \beta}^-$ \hspace{1cm} (− extensionality)
- $(-6)$\hspace{1cm} $K \subseteq Pr(K_{\alpha \rightarrow \beta}^- \cup \{\alpha \Rightarrow \beta\})$ \hspace{1cm} (− recovery)

Theorem [Casini & Meyer (2017)]

A contraction operator – for preferential entailment $Pr$ satisfies $(-1) - (-6)$ iff it is a partial meet contraction operator.
Preferential revision

Monotonic Core \( Pr. \) Postulates for Revision • (consistency preservation)

The postulates for revision for consistency preservation are as follows:

• (1) \( K_{\alpha \Rightarrow \beta} = Pr(K^*_{\alpha \Rightarrow \beta}) \)  (• closure)

• (2) \( K^*_{\alpha \Rightarrow \beta} \subseteq Pr(K \cup \{ \alpha \Rightarrow \beta \}) \)  (• inclusion)

• (3) If \( \top \Rightarrow \bot \notin Pr(K \cup \{ \alpha \Rightarrow \beta \}) \), then \( Pr(K \cup \{ \alpha \Rightarrow \beta \}) \subseteq K^*_{\alpha \Rightarrow \beta} \)  (• vacuity)

• (4) \( \alpha \Rightarrow \beta \in K^*_{\alpha \Rightarrow \beta} \)  (• success)

• (5) If \( \alpha \Rightarrow \beta \equiv_{Pr} \alpha' \Rightarrow \beta' \), then \( K^*_{\alpha \Rightarrow \beta} = K^*_{\alpha' \Rightarrow \beta'} \)  (• extensionality)

• (6) If \( \top \Rightarrow \bot \notin Pr(\alpha \Rightarrow \beta) \), then \( \top \Rightarrow \bot \notin Pr(K^*_{\alpha \Rightarrow \beta}) \)  (• consistency)

• (+) \( K^*_{\alpha \Rightarrow \beta} = Pr(K^*_{\top \Rightarrow \alpha \Rightarrow \beta} \cup \{ \alpha \Rightarrow \beta \}) \)  (• extra)
Preferential revision

Levi-style Identity for consistency preservation:

\[ K_\alpha \Rightarrow \beta := Pr(K_{\top} \Rightarrow \alpha \land \lnot \beta \cup \{ \alpha \Rightarrow \beta \}) \quad (1) \]

Theorem [Casini & Meyer (2017)]

A revision operator \( \bullet \) for preferential entailment \( Pr \) satisfies \((\bullet \ 1) - (\bullet \ 6)\) and \((\bullet +)\) iff it can be defined, via \((1)\), from a contraction operator satisfying the postulates \((- \ 1) - (-6)\)
Monotonic Core \( Pr. \)
Postulates for Revision \( \circ \) (coherence preservation)

The postulates for revision for coherence preservation are as follows:

\( \circ 1 \) \( K_{\alpha \Rightarrow \beta}^\circ = Pr(K_{\alpha \Rightarrow \beta}) \)  
(\( \circ \) closure)

\( \circ 2 \) \( K_{\alpha \Rightarrow \beta}^\circ \subseteq Pr(K \cup \{\alpha \Rightarrow \beta\}) \)  
(\( \circ \) inclusion)

\( \circ 3 \) \textbf{If} \( \alpha \Rightarrow \bot \notin Pr(K \cup \{\alpha \Rightarrow \beta\}) \textbf{, then} \ Pr(K \cup \{\alpha \Rightarrow \beta\}) \subseteq K_{\alpha \Rightarrow \beta}^\circ \)  
(\( \circ \) vacuity)

\( \circ 4 \) \( \alpha \Rightarrow \beta \in K_{\alpha \Rightarrow \beta}^\circ \)  
(\( \circ \) success)

\( \circ 5 \) \textbf{If} \( \alpha \Rightarrow \beta \equiv_{Pr} \alpha' \Rightarrow \beta' \textbf{, then} \ K_{\alpha \Rightarrow \beta}^\circ = K_{\alpha' \Rightarrow \beta'}^\circ \)  
(\( \circ \) extensionality)

\( \circ 6 \) \textbf{If} \( \alpha \Rightarrow \bot \notin Pr(\alpha \Rightarrow \beta) \textbf{, then} \ \alpha \Rightarrow \bot \notin Pr(K_{\alpha \Rightarrow \beta}^\circ) \)  
(\( \circ \) coherence)
Preferential revision

Levi-style Identity for consistency preservation:

\[ K_\alpha \Rightarrow_\beta := Pr(K_\alpha \Rightarrow \neg \beta \cup \{\alpha \Rightarrow \beta\}) \]  \hspace{1cm} (2)

Theorem [Casini et Al. (2018)]

A revision operator \( \circ \) for preferential entailment \( Pr \) satisfies \((\circ 1) - (\circ 6)\) iff it can be defined, via (2), from a contraction operator satisfying the postulates \((- 1) - (- 6)\)
BR for preferential conditionals

We have characterised contraction and revision for the monotonic core.

In order to characterise revision w.r.t. a \( Cn \)-theory \( K \), we need to keep track and refer to it’s monotonic core \( (K_p) \):

\[ K = Cn(K_p) \]

- We want to add \( A \Rightarrow B \) to \( K \). An actual revision needs to be done only if there is a conflict with the monotonic core \( K_p \).

Revision of a non-monotonic theory would always keep track of the theory and its monotonic core.
BR for preferential conditionals

Given $Cn$-theory $K$, we need to keep track and refer to it’s monotonic core ($K_P$)

Non-monotonic Closure $Cn$. Postulates for Revision $\odot$ (consistency preservation)

The postulates for revision for consistency preservation are as follows:

- (⊙ 1) $K_{\alpha \Rightarrow \beta}^\odot = Cn(K_{\alpha \Rightarrow \beta}^\odot)$ (⊙ closure)
- (⊙ 2) $\exists K' \text{ s.t. } Cn(K') = Cn(K_{\alpha \Rightarrow \beta}^\odot)$ and $K' \subseteq Pr(K_P \cup \{\alpha \Rightarrow \beta\})$ (⊙ generator inclusion)
- (⊙ 3) If $\top \not\Rightarrow \bot \not\in Pr(K_P \cup \{\alpha \Rightarrow \beta\})$, then $Cn(K_P \cup \{\alpha \Rightarrow \beta\}) \subseteq K_{\alpha \Rightarrow \beta}^\odot$ (⊙ vacuity)
- (⊙ 4) $\alpha \Rightarrow \beta \in K_{\alpha \Rightarrow \beta}^\odot$ (⊙ success)
- (⊙ 5) If $\alpha \Rightarrow \beta \equiv_{Pr} \alpha' \Rightarrow \beta'$, then $K_{\alpha \Rightarrow \beta}^\odot = K_{\alpha' \Rightarrow \beta'}^\odot$ (⊙ extensionality)
- (⊙ 6) If $\top \not\Rightarrow \bot \not\in Pr(\alpha \Rightarrow \beta)$, then $\top \not\Rightarrow \bot \not\in Pr(K_{\alpha \Rightarrow \beta}^\odot)$ (⊙ consistency)
BR for preferential conditionals

Non-monotonic Closure $Cn$. Postulates for Revision $\otimes$ (coherence preservation)

The postulates for revision for consistency preservation are as follows:

1. $K\alpha \Rightarrow \beta = Cn(K\alpha \Rightarrow \beta)$ (closure)
2. $\exists K' \text{ s.t. } Cn(K') = Cn(K\alpha \Rightarrow \beta) \text{ and } K' \subseteq Pr(K_p \cup \{\alpha \Rightarrow \beta\})$ (generator inclusion)
3. If $\alpha \Rightarrow \bot \notin Pr(K_p \cup \{\alpha \Rightarrow \beta\})$, then $Cn(K_p \cup \{\alpha \Rightarrow \beta\}) \subseteq K\alpha \Rightarrow \beta$ (vacuity)
4. $\alpha \Rightarrow \beta \in K\alpha \Rightarrow \beta$ (success)
5. If $\alpha \Rightarrow \beta \equiv_{Pr} \alpha' \Rightarrow \beta'$, then $K\alpha \Rightarrow \beta = K\alpha' \Rightarrow \beta'$ (extensionality)
6. If $\alpha \Rightarrow \bot \notin Pr(\alpha \Rightarrow \beta)$, then $\alpha \Rightarrow \bot \notin Pr(K\alpha \Rightarrow \beta)$ (coherence)
BR for preferential conditionals

Theorem [Casini & Meyer (2017)]

A revision operator $\odot$ for suprapreferential entailment $Cn$ satisfies $(\odot 1) - (\odot 6)$ iff there is a preferential revision operator $\bullet$ satisfying the postulates $(\bullet 1) - (\bullet 6)$ s.t.

$$K_{\alpha \Rightarrow \beta}^\odot = Cn(K_{p \alpha \Rightarrow \beta} \bullet)$$

Theorem [Casini & Meyer (2017)]

A revision operator $\otimes$ for suprapreferential entailment $Cn$ satisfies $(\otimes 1) - (\otimes 6)$ iff there is a preferential revision operator $\circ$ satisfying the postulates $(\circ 1) - (\circ 6)$ s.t.

$$K_{\alpha \Rightarrow \beta}^\otimes = Cn(K_{p \alpha \Rightarrow \beta} \circ)$$

The semantic characterisation of the above operations will be presented in [Casini et Al. (2018)]
Example

We have the following KB $B$, that is closed by a supra-preferential closure operator $Cn$:

$$\text{horse} \Rightarrow \text{tall}; \quad \text{horse} \Rightarrow \text{black};$$

$$\text{horse} \Rightarrow \text{live . in . farm}$$

Let’s consider some new pieces of information:

- $\text{horse} \Rightarrow \neg (\text{tall} \land \text{live . in . farm})$
- $\text{horse} \land \text{black} \Rightarrow \neg \text{tall}$
- $\text{horse} \land \text{brown} \Rightarrow \neg \text{tall}$
Example

We have the following KB $B$, that is closed by a supra-preferential closure operator $Cn$:

$$\text{horse} \Rightarrow \text{tall}; \quad \text{horse} \Rightarrow \text{black};$$

$$\text{horse} \Rightarrow \text{live} \cdot \text{in} \cdot \text{farm}$$

Let’s consider some new pieces of information:

- $\text{horse} \Rightarrow \neg (\text{tall} \land \text{live} \cdot \text{in} \cdot \text{farm})$
- $\text{horse} \land \text{black} \Rightarrow \neg \text{tall}$
- $\text{horse} \land \text{brown} \Rightarrow \neg \text{tall}$

How do we manage the introduction of each of these pieces of information, starting from $B$?
Example

Knowledge base $B$:  

\[ \text{horse} \Rightarrow \text{tall}; \quad \text{horse} \Rightarrow \text{black}; \]
\[ \text{horse} \Rightarrow \text{live . in . farm} \]

Preferential Properties (defining the monotonic core):

- (REF)  $\alpha \Rightarrow \alpha$  Refexivity
- (CT)  $\frac{\alpha \Rightarrow \beta \quad \alpha \land \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$  Cut (Cumulative Trans.)
- (CM)  $\frac{\alpha \Rightarrow \beta \quad \alpha \Rightarrow \gamma}{\alpha \land \beta \Rightarrow \gamma}$  Cautious Monotony
- (LLE)  $\frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\beta \Rightarrow \gamma}$  Left Logical Equival.
- (RW)  $\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$  Right Weakening
- (OR)  $\frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \lor \beta \Rightarrow \gamma}$  Left Disjunction

New conditionals:

- $\text{horse} \Rightarrow \neg (\text{tall} \land \text{live . in . farm})$
- $\text{horse} \land \text{black} \Rightarrow \neg \text{tall}$
- $\text{horse} \land \text{brown} \Rightarrow \neg \text{tall}$
Contraction

We have investigated the revision of a non-monotonic conditional closure $Cn$.

What about contraction w.r.t. a non-monotonic conditional closure $Cn$?

We have characterised contraction w.r.t. the monotonic part $Pr(.)$, but what happens if I want to contract $\alpha \Rightarrow \beta$ from a theory $K$ closed under $Cn(.)$?
Contraction

Two possibilities:

1. $\alpha \Rightarrow \beta$ is in the monotonic core of $K$.
   We need to apply the preferential contraction to the monotonic core $K_p$ of $K$: $(K_p)_{\alpha \Rightarrow \beta}$

\[
K^\alpha_{\alpha \Rightarrow \beta} = Cn(K_p^{-}_{\alpha \Rightarrow \beta})
\]
Contraction

2. \( \alpha \Rightarrow \beta \) is in the defeasible part of \( K \).
   We can eliminate information from the monotonic core \( K_p \) of \( K \), but there is an alternative. Check the previous example:

\[
B' = \{ v \Rightarrow n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \}
\]

Assuming our non-monotonic machinery is well-behaved,

- from \( B' \) we can conclude that avian red blood cells presumably have a nucleus (\( a \Rightarrow n \))

- But we are informed that avian red blood cells usually do not have a nucleus (\( a \Rightarrow \neg n \)). Since \( a \Rightarrow n \) was a presumptive conclusion, the non-monotonic entailment relation should take care of eliminating such a conclusion once faced with conflicting evidence (\( a \Rightarrow \neg n \)).

- The introduction of \( a \Rightarrow \neg n \) should correspond to a simple addition:

\[
B'' = \{ v \Rightarrow n, a \Rightarrow \neg n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \}
\]
Contraction

Let’s rephrase it:

\[ B' = \{ v \Rightarrow n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \} \]

Assuming our non-monotonic machinery is well-behaved,

• from \( B' \) we can conclude that avian red blood cells presumably have a nucleus (\( a \Rightarrow n \))

• But we want to contract \( a \Rightarrow n \) from \( Cn(B') \). We decide to add conflicting information to our KB (\( a \Rightarrow \neg n \)), in order to prevent the system from concluding

\[ B'' = \{ v \Rightarrow n, a \Rightarrow \neg n, a \rightarrow v, m \rightarrow v, m \rightarrow \neg n \} \]

• The introduction of \( a \Rightarrow \neg n \) should prevent the derivation of \( a \Rightarrow n \)
Contraction

We can make two observations from the above example:

1. The is the possibility of contracting information by adding some new piece of information.

2. The same operation (e.g., the addition of \( a \implies \neg n \)), could be interpreted as a revision or a contraction, depending on the goal (i.e., the introduction of \( a \implies \neg n \) or the elimination of \( a \implies n \)).
The investigation of non-monotonic contraction in the conditional framework has to be done.

It is instead at the base of the approach to revision for logic programs in [Zhuang et Al. (2016)].

Disjunctive logic programs are based on rules of the form:

\[ a_1, \ldots, a_m \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_0 \]
Consider the following program:

- \( \text{Teach}(\text{John}) \leftarrow \text{Prof}(\text{John}), \textbf{not} \text{Admin}(\text{John}) \)
- \( \text{Prof}(\text{John}) \leftarrow \)

From this we conclude \( \{ \text{Prof}(\text{John}), \text{Teach}(\text{John}) \} \)
We are informed that

- \( \leftarrow \text{Teach}(\text{John}) \)
that is in \textbf{conflict} with the previous program (no answer set).
Consider the following program:

- \textit{Teach}(John) \leftarrow \textit{Prof}(John), \textbf{not} \textit{Admin}(John)
- \textit{Prof}(John) \leftarrow

From this we conclude \{\textit{Prof}(John), \textit{Teach}(John)\}
We are informed that

- \leftarrow \textit{Teach}(John)
  that is in \textbf{conflict} with the previous program (no answer set).

We can \textbf{fix} the situation in two ways:

- We \textbf{eliminate} some rule in the program, or
- We \textbf{add} \textit{Admin}(John) \leftarrow to the program.
ASP

[Zhuang et Al. (2016)] characterise belief change in the framework of grounded disjunctive logic programs defining an operator $P \star Q$ s.t.:

$P$ and $Q$ are two programs, and $P \star Q$ gives back a consistent program containing $Q$.

In case of conflict, either $P$ is weakened, or more rules are added.

If the latter solution is impossible, it means that the conflict between $P$ and $Q$ is a monotonic inconsistency.
Essential Bibliography

Preferential conditionals:


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AGM vs. conditionals:


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Revision of Conditional KBs:


Revision in ASP: