Description Logics

Exercises 1

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Concept Language

1. Let A and B be atomic concept names, let r and s be role names (attributes), and let a and b be individual names. For each of the following expressions, decide whether it is an $\mathcal{ALC}$ concept or not. In the negative case, explain why.

- $A \neg B$
- $\neg\neg A \sqcap B$
- $A \sqcap \neg A \sqcup \exists s.\forall r.\bot$
- $A \sqcap (\neg A \sqcup B)$
- $A \sqcup B \sqcap \neg A$
- $\neg A \sqcup \sqcap B$
- $(\neg A) \sqcup \sqcap B$
- $\neg A \sqcup (\sqcap B)$
- $\exists A.B$
- $\forall A.\neg r$
- $\exists r.(A \sqcap \neg B) \sqcup \forall s.\neg A$
- $\forall r.\exists s.A \sqcap \exists r.\top$
- $\neg r \sqcup s$
- $\forall (r \sqcap s).A$
- $\forall r.(A \sqcap B)$
- $\exists B.(A \sqcap r \sqcup \neg s)$
- $\exists r.A \sqcap \exists s.B$
- $\exists r.(A \sqcap \exists s.B)$
- $r \sqcup s$
- $\neg\exists r.A \sqcup \neg\forall s.\neg A \sqcap B$
- $A \sqcap \leq 1 r.A$
- $\forall r^{-}.A$
- $\{a, b\}$
- $a \sqcap b$
- $\exists r.a$
- $\exists r.(a)$
- $\exists r.\{a\}$
- $\forall (r \circ s).A$
- $\forall r.\forall s.A$
- $\neg(A \sqcap B) \equiv (\neg A \sqcup \neg B)$
- $(A \sqcap B) \to (B \sqcup A)$
- $(A \to B) \to (A \sqcap B \to B)$
- $(A \to B) \equiv (\neg A \sqcup B)$
- $\neg\neg A \equiv A$
- $\forall r.(A \sqcap \exists s.(B \sqcup \forall r.(A \sqcap \exists s.(B \sqcup \forall r.(A \sqcap \exists s.\top)))))$
- $\exists r.(A \sqcup \forall s.(B \sqcap \exists r.(A \sqcup \forall s.(B \sqcap \exists r.(A \sqcup \forall s.\bot)))))$

Semantics

2. Let A and B be atomic concept names, let r and s be role names (attributes), and let a and b be individual names. Let $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ be an interpretation such that:

$$\Delta^{\mathcal{I}} := \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$$

$$A^{\mathcal{I}} := \{x_1, x_3, x_5, x_7, x_9\}$$
$$B^{\mathcal{I}} := \{x_1, x_2, x_3\}$$
$$r^{\mathcal{I}} := \{(x_1, x_2), (x_2, x_3), (x_1, x_3), (x_7, x_8), (x_8, x_9), (x_7, x_9)\}$$
$$s^{\mathcal{I}} := \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_5), (x_9, x_8), (x_8, x_7), (x_7, x_6)\}$$
$$a^{\mathcal{I}} := x_1, \quad b^{\mathcal{I}} = x_2$$

(a) Represent $\mathcal{I}$ graphically.

(b) Determine:

- $(\neg B)^{\mathcal{I}}$
- $(\neg A)^{\mathcal{I}}$
- $(A \sqcup B)^{\mathcal{I}}$
- $(A \sqcap B)^{\mathcal{I}}$
- $(\neg A \sqcap \neg B)^{\mathcal{I}}$
- $(\neg B)^{\mathcal{I}}$
- $(\exists r.A)^{\mathcal{I}}$
- $(\exists r.\neg B)^{\mathcal{I}}$
- $(\exists r.\top)^{\mathcal{I}}$
- $(\forall r.\bot)^{\mathcal{I}}$
- $(\exists r.\bot)^{\mathcal{I}}$
- $(\forall r.\top)^{\mathcal{I}}$
- $(\exists r.\exists s.\top)^{\mathcal{I}}$
- $(\exists s.A)^{\mathcal{I}}$
- $(\exists s.\neg B)^{\mathcal{I}}$
- $(\exists s.\top)^{\mathcal{I}}$
- $(\forall s.\bot)^{\mathcal{I}}$
- $(\forall s.\exists r.A)^{\mathcal{I}}$
- $(\forall s.\exists r.\neg A)^{\mathcal{I}}$
- $(\exists s.\forall r.\bot)^{\mathcal{I}}$
- $(\exists s.\exists r.\forall \top)^{\mathcal{I}}$
- $(\neg \exists r.\forall s.\exists s.\neg(A \sqcup B))^{\mathcal{I}}$
- $(\forall r.\exists s.\exists r.\exists s.\bot)^{\mathcal{I}}$
- $(A \sqcup B \sqcap \forall r.A)^{\mathcal{I}}$
- $(B \sqcup A \sqcap \exists s.\neg B)^{\mathcal{I}}$
- $(\forall r.(A \sqcap \exists s.(B \sqcup \forall r.(A \sqcap \exists s.(B \sqcup \forall r.(A \sqcap \exists s.\top))))))^{\mathcal{I}}$

3. Let $\mathsf{N}_{\mathscr{C}} := \{A_1, A_2, A_3\}$, $\mathsf{N}_{\mathscr{R}} := \{r_1, r_2, r_3\}$ and $\mathsf{N}_{\mathscr{I}} := \{a_1, a_2, a_3, a_4, a_5\}$. Let $\mathcal{I}_1 := \langle \Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1} \rangle$ and $\mathcal{I}_2 := \langle \Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2} \rangle$ be interpretations such that:

$$\Delta^{\mathcal{I}_1} := \{x_1, x_2, x_3, x_4, x_5\}$$
$$A^{\mathcal{I}_1} := \{x_1, x_2, x_3\}$$
$$B^{\mathcal{I}_1} := \{x_2, x_4, x_5\}$$
$$r_1^{\mathcal{I}_1} := \{(x_2, x_1), (x_1, x_3), (x_4, x_5)\}$$
$$r_2^{\mathcal{I}_1} := \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5)\}$$
$$r_3^{\mathcal{I}_1} := \{(x_2, x_4), (x_4, x_4)\}$$
$$a_1^{\mathcal{I}_1} := x_5, a_2^{\mathcal{I}_1} := x_4, a_3^{\mathcal{I}_1} := x_3,$$
$$a_4^{\mathcal{I}_1} := x_2, a_5^{\mathcal{I}_1} := x_1$$

$$\Delta^{\mathcal{I}_2} := \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$$
$$A^{\mathcal{I}_2} := \{y_1, y_2, y_3, y_5, y_7\}$$
$$B^{\mathcal{I}_2} := \{y_4, y_5, x_7\}$$
$$r_1^{\mathcal{I}_2} := \{(y, y) \mid y \in \Delta^{\mathcal{I}_2}\}$$
$$r_2^{\mathcal{I}_2} := \{(y_i, y_{i+2}) \mid 1 \leq i \leq 5\}$$
$$r_3^{\mathcal{I}_2} := \{(y_j, y_i) \mid (y_i, y_j) \in r_2^{\mathcal{I}_2}\}$$
$$a_1^{\mathcal{I}_2} := y_2, a_2^{\mathcal{I}_2} := y_3, a_3^{\mathcal{I}_2} := y_4,$$
$$a_4^{\mathcal{I}_2} := y_5, a_5^{\mathcal{I}_2} := y_7$$

(a) Represent $\mathcal{I}_1$ e $\mathcal{I}_2$ graphically.

(b) Determine:

- $\top^{\mathcal{I}_1}$
- $\top^{\mathcal{I}_2}$
- $(A \sqcap B)^{\mathcal{I}_1}$
- $(A \sqcap B)^{\mathcal{I}_2}$
- $(\neg A \sqcap B)^{\mathcal{I}_1}$
- $(A \sqcap \neg B)^{\mathcal{I}_2}$
- $(\exists r_1.A)^{\mathcal{I}_1}$
- $(\exists r_1.\neg B)^{\mathcal{I}_2}$
- $(\exists r_2.\top)^{\mathcal{I}_2}$
- $(\forall r_3.\bot)^{\mathcal{I}_1}$
- $(\exists r_3.\bot)^{\mathcal{I}_1}$
- $(\forall r_2.\top)^{\mathcal{I}_2}$
- $(\neg A \sqcap \exists r_3.A)^{\mathcal{I}_1}$
- $(A \sqcup B \sqcap \forall r_2.A)^{\mathcal{I}_2}$
- $(B \sqcup A \sqcap \exists r_1.\neg B)^{\mathcal{I}_1}$
- $(\forall r_1.\exists r._2 A)^{\mathcal{I}_1}$
- $(\exists r_2.\forall r_3.\bot)^{\mathcal{I}_2}$
- $(\exists r_3.\exists r_1.\top)^{\mathcal{I}_1}$
- $(\neg \exists r_1.\forall r_2.\exists r_3.\neg(A \sqcup B))^{\mathcal{I}_2}$
- $(\forall r_3.\exists r_2.\exists r_1.\exists r_2.\bot)^{\mathcal{I}_1}$

4. Let $\mathsf{N}_{\mathscr{C}} = \{\mathsf{Man}, \mathsf{Woman}, \mathsf{Husband}, \mathsf{Wife}, \mathsf{Parent}, \mathsf{Grandparent}\}$, $\mathsf{N}_{\mathscr{R}} = \{\mathsf{marriedTo}, \mathsf{hasChild}\}$ and $\mathsf{N}_{\mathscr{I}} = \{\mathsf{john}, \mathsf{mary}, \mathsf{alice}, \mathsf{pete}, \mathsf{anne}\}$. Define three different interpretations for $\mathsf{N}_{\mathscr{C}} \cup \mathsf{N}_{\mathscr{R}} \cup \mathsf{N}_{\mathscr{I}}$.

5. Let C, D be complex $\mathcal{ALC}$ concepts and let r be a role name.

(a) Show that the following equalities hold for every interpretation $\mathcal{I}$.

- $(\neg\neg C)^{\mathcal{I}} = C^{\mathcal{I}}$
- $(\neg(C \sqcap D))^{\mathcal{I}} = (\neg C \sqcup \neg D)^{\mathcal{I}}$
- $(\neg(C \sqcup D))^{\mathcal{I}} = (\neg C \sqcap \neg D)^{\mathcal{I}}$
- $(\neg \forall r.C)^{\mathcal{I}} = (\exists r.\neg C)^{\mathcal{I}}$
- $(\neg \exists r.C)^{\mathcal{I}} = (\forall r.\neg C)^{\mathcal{I}}$
- $\bot^{\mathcal{I}} = (\exists r.\bot)^{\mathcal{I}}$
- $\top^{\mathcal{I}} = (\forall r.\top)^{\mathcal{I}}$
- $(\neg C \sqcap (C \sqcup D))^{\mathcal{I}} = D^{\mathcal{I}}$

(b) Show that the following equalities do **not** hold (for every interpretation $\mathcal{I}$).

- $(\exists r.\forall r.C)^{\mathcal{I}} = (\forall r.\exists r.C)^{\mathcal{I}}$
- $(\exists r.\top)^{\mathcal{I}} = (\exists r.C)^{\mathcal{I}}$
- $(\forall r.\bot)^{\mathcal{I}} = (\forall r.C)^{\mathcal{I}}$
- $(\exists r.\top)^{\mathcal{I}} = (\exists r.\exists r.\top)^{\mathcal{I}}$
- $(\forall r.\forall r.\bot)^{\mathcal{I}} = (\forall r.\bot)^{\mathcal{I}}$

(c) For each of the following equalities, decide whether it holds or not. If yes, then prove it; else show a counter-example.

- $(\forall r.(C \sqcap D))^{\mathcal{I}} = (\forall r.C \sqcap \forall r.D)^{\mathcal{I}}$
- $(\forall r.(C \sqcup D))^{\mathcal{I}} = (\forall r.C \sqcup \forall r.D)^{\mathcal{I}}$
- $(\exists r.(C \sqcap D))^{\mathcal{I}} = (\exists r.C \sqcap \exists r.D)^{\mathcal{I}}$
- $(\exists r.(C \sqcup D))^{\mathcal{I}} = (\exists r.C \sqcup \exists r.D)^{\mathcal{I}}$
- $(\forall r.C \sqcap \exists r.\neg C)^{\mathcal{I}} = (\bot)^{\mathcal{I}}$
- $(\forall r.C \sqcap \forall r.\neg C)^{\mathcal{I}} = (\top)^{\mathcal{I}}$
- $(\forall r.\bot \sqcap \forall r.C)^{\mathcal{I}} = (\forall r.\bot)^{\mathcal{I}}$
- $(\exists r.\top \sqcup \exists r.C)^{\mathcal{I}} = (\exists r.(C \sqcup D))^{\mathcal{I}}$